402. Proposed by C. N. SCHMALL, New York City.

If (x, y) be a double point on the curve $u \equiv f(x, y) = 0$, show that (1) the two branches of the curve will cut orthogonally if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0;$$

and (2), if this point be made the origin, then the equation of the tangents to the branches will be

$$(y'^2 - x'^2)\frac{\partial^2 u}{\partial x^2} + 2x'y'\frac{\partial^2 u}{\partial x \partial y} = 0$$

where (x', y') are the current coördinates of points on the tangents.

SOLUTION BY C. E. DIMICK, New London, Connecticut.

If (x, y) be a double point of the curve $u \equiv f(x, y) = 0$, $\partial u/\partial x = \partial u/\partial y = 0$ and the two values of dy/dx at the double point must satisfy the quadratic

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{dy}{dx} + \frac{\partial^2 u}{\partial y^2}\left(\frac{dy}{dx}\right)^2 = 0.$$

(Todhunter's Diff. Calc., pages 319, 320.)

If the two tangents are perpendicular, the product of their slopes is -1, and since the product of the roots of the quadratic $a + bx + cx^2 = 0$ is a/c we have

$$\frac{\partial^2 u}{\partial x^2}\Big/\frac{\partial^2 u}{\partial y^2} = -1 \qquad \text{or} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The equations of the two tangents will be $y - y_1 = l_1(x - x_1)$, and $y - y_1 = l_2(x - x_1)$, where l_1 and l_2 are the roots of the quadratic given above. Transposing and multiplying, the equation of the two tangents taken together will be

$$(y - y_1)^2 - (x - x_1)(y - y_1)(l_1 + l_2) + l_1 l_2 (x - x_1)^2 = 0,$$

which becomes on substituting the values of the sum and product of the roots of the quadratic

$$(y - y_1)^2 + 2(x - x_1)(y - y_1)\frac{\partial^2 u}{\partial x \partial y}\Big/\frac{\partial^2 u}{\partial y^2} + (x - x_1)^2\frac{\partial^2 u}{\partial x^2}\Big/\frac{\partial^2 u}{\partial y^2} = 0,$$

which, upon transforming to (x_1, y_1) as origin, becomes

$$y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial x^2} = 0.$$

But $\partial^2 u/\partial x^2 = -\partial^2 u/\partial y^2$ as the tangents are perpendicular.

Hence the equation reduces to

$$(y^2 - x^2)\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 0,$$

the minus sign as given in the problem being incorrect.

Also solved by I. A. BARNETT, C. K. ROBBINS, GRACE M. BAREIS, J. A. BULLARD, F. M. MORGAN, and H. L. AGARD.

MECHANICS.

304. Proposed by B. F. FINKEL, Drury College.

A spherical shell, inner radius r and outer radius R, has within it a perfectly smooth solid sphere of the same material and with radius $r_1 < r$. If the inner surface of the spherical shell is also perfectly smooth, determine the motion, after the time t, of the shell and sphere down a rough inclined plane, inclination α.